

AD A099397

LEVEL

12

DTIC  
SELECTED  
MAY 27 1981  
C

DEPARTMENT OF STATISTICS

The Ohio State University

OSU

COLUMBUS, OHIO

Best Available Copy

DTIC FILE COPY

81 5 21 037

12

ENTROPY-BASED RANDOM NUMBER EVALUATION\*

by

Edward J. Dudewicz,  
Edward C. van der Meulen,  
M. G. SriRam, and  
K. W. Teoh

DTIC  
SELECTED  
MAY 27 1981

Technical Report No. 232  
Department of Statistics  
The Ohio State University  
Columbus, Ohio 43210

April 1981

\* Research supported by Office of Naval Research Contract  
No. N00014-78-C-0543.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

(14) TR-232

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Report No. 232	2. GOVT ACCESSION NO. AD-A099397	3. REPORT'S CATALOG NUMBER (9)
4. TITLE (and Subtitle) ENTROPY-BASED RANDOM NUMBER EVALUATION.		5. TYPE OF REPORT & PERIOD COVERED Technical Report
6. PERFORMING ORG. REPORT NUMBER		7. AUTHOR(s) Edward J./Dudewicz Edward C. van der/Meulen M. G./SriRam K. W./Teoh
8. CONTRACT OR GRANT NUMBER(s)		9. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 042-403
10. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Department of the Navy Arlington, Virginia 22217		11. REPORT DATE Apr 1981
12. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		13. NUMBER OF PAGES iii + 43
14. DISTRIBUTION STATEMENT (of this Report) Approval for public release; distribution unlimited.		15. SECURITY CLASS. (of this report) Unclassified
16. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Key Words and Phrases: Random number generators; entropy-based tests for uniformity; asymptotic normality; Monte Carlo estimates of percentage points and power; goodness of fit tests; evaluation and comparison; chi-square entropy tests.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Recent work has shown how to test a simple hypothesis of uniformity on the interval (0,1) by using estimates of entropy. In this paper we use Monte Carlo methods to extend previous tables of critical points and power for such entropy tests to the large sample sizes likely to be desirable when evaluating the output of one or more random number generators. A comparison with asymptotic critical points and power is made. The results are used to evaluate a number of commonly used random number		

DD FORM 1 JAN 73 1473

406331

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

20. generators. At least one random number generator judged "good" by the spectral test (multiplier 5\*\*15) is found unsuitable for use.

Accession For	
NTIS GRA&I	<input checked="checked" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By _____	
Distribution/ _____	
Availability Codes	
Dist	Avail and/or Special
A	

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

## ENTROPY-BASED RANDOM NUMBER EVALUATION\*

by

Edward J. Dudewicz,  
Edward C. van der Meulen,  
M. G. SriRam, and  
K. W. Teoh

### ABSTRACT

Recent work has shown how to test a simple hypothesis of uniformity on the interval (0,1) by using estimates of entropy. In this paper we use Monte Carlo methods to extend previous tables of critical points and power for such entropy tests to the large sample sizes likely to be desirable when evaluating the output of one or more random number generators. A comparison with asymptotic critical points and power is made. The results are used to evaluate a number of commonly used random number generators. At least one random number generator judged "good" by the spectral test (multiplier  $5^{**15}$ ) is found unsuitable for use.

---

\* Research supported by Office of Naval Research Contract No. N00014-78-C-0543.

## 1. INTRODUCTION

An estimator of entropy was proposed by Vasicek (1976) and studied by Dudewicz and van der Meulen (1979), who showed how this estimator could be used to test whether a random sample comes from the uniform distribution on  $(0,1)$  and gave Monte Carlo and asymptotic evaluations of the percentage points of the distribution of the estimator. In this paper we utilize this test for uniformity to evaluate random number generators. A comparison of asymptotic with Monte Carlo for percent points and power is also given.

The differential entropy (or entropy) of a random variable  $X$  with density function  $f$  is defined as

$$H(X) = - \int_{-\infty}^{\infty} f(x) \log f(x) dx \quad (1.1)$$

and has the properties that if  $X \in [0,1]$  w.p. 1 then

$$H(X) \leq 0, \quad (1.2)$$

and among all densities  $f$  concentrated on  $[0,1]$  the uniform  $f_0$  maximizes  $H(f)$  to

$$H(f_0) = 0. \quad (1.3)$$

The following two definitions were given by Dudewicz and van der Meulen (1979), and use the above.

Definition (1.4). Two densities  $f_1$  and  $f_2$  are said to be entropy-distinguishable if  $H(f_1) \neq H(f_2)$ .

Definition (1.5). A density  $f^*$  is said to be entropy-unique (or e-unique) in a class  $\mathcal{C}$  of densities if  $f^* \in \mathcal{C}$  and  $\nexists f \in \mathcal{C}, f \neq f^*$ , such that  $H(f) = H(f^*)$ .

By property (1.3), we know that in the class  $\mathcal{C}$  of densities on  $(0,1)$ ,  $f_0$  is e-unique. This fact will be central below.

Now let  $X_1, \dots, X_n$  be a random sample from an absolutely continuous distribution  $F$  with density  $f$  and let  $X_{(1)}, \dots, X_{(n)}$  be the order statistics. The estimator of  $H(f)$  (Vasicek (1976)) is

$$H_{m,n} = n^{-1} \sum_{i=1}^n \log \left\{ \frac{n}{2m} [X_{(i+m)} - X_{(i-m)}] \right\} \quad (1.6)$$

where  $1 \leq m < n/2$ ,  $X_{(j)} = X_{(1)}$  for  $j < 1$ , and  $X_{(j)} = X_{(n)}$  for  $j > n$ .

Dudewicz and van der Meulen (1979) proposed a test for uniformity based on the e-uniqueness of  $f_0$ . We are concerned with this test and so describe it here.

Let  $X_1, \dots, X_n$  be a random sample from an absolutely continuous distribution  $F$  with density  $f$  concentrated on  $[0,1]$ . Let  $f_0$  denote the  $U(0,1)$  density. The level  $\alpha$  test rejects  $H_0 : f = f_0$  in favor of  $H_A : f \neq f_0$  if and only if

$$H_{m,n} \leq H_{\alpha}^*(m,n) \quad (1.7)$$

where  $H_{\alpha}^*(m,n)$  is the  $100\alpha$  percentile point of the distribution of  $H_{m,n}$  under  $f_0$ .

By the e-uniqueness of  $f_0$  among all densities concentrated on  $[0,1]$  and by the consistency of the estimator (Vasicek (1976)), it follows that the above test is consistent against all alternatives  $f$  on  $[0,1]$ . Further, in Dudewicz and van der Meulen (1979) it is shown that if  $f$  is concentrated on  $[0,1]$  then, w.p. 1,  $H_{m,n} \leq 0$ .

In Dudewicz and van der Meulen (1979), the above test is studied in detail using both analytical and Monte Carlo techniques. The latter are necessary since the form of the distribution of  $H_{m,n}$  appears to be analytically intractable. From Dudewicz and van der Meulen (1979) we know

Theorem 1.8. Under  $H_0$ , if  $m = o(n^{1/3-\delta})$ ,  $\delta > 0$ , the quantities

$$(6mn)^{1/2} [H_{m,n} - \log(\frac{n}{2m}) + \log(n+1) + \gamma - R(1, 2m-1)] \quad (1.9)$$

and

$$(6mn)^{1/2} [H_{m,n} + \log(2m) + \gamma - R(1, 2m-1)] \quad (1.10)$$

are asymptotically  $N(0,1)$  as  $n \rightarrow \infty$ .



Here  $\gamma$  is Euler's constant,  $\gamma = 0.5772\dots$  and, for  $j \leq k$ ,  $R(j, k) = k^{-1} + (k-1)^{-1} + \dots + j^{-1}$ . Using (1.9) we then have that

$$\hat{H}_\alpha^*(m, n) = \Phi^{-1}(\alpha)(6mn)^{-1/2} - \log(2m) - \gamma + R(1, 2m-1) - \log\left(\frac{n+1}{n}\right) \quad (1.11)$$

is an asymptotic approximation to the  $100\alpha$  percentile point of the distribution of  $H_{m,n}$  where  $\Phi(\cdot)$  is the standard normal distribution function.

For asymptotic power, Dudewicz and van der Meulen (1979) show

Theorem 1.12. For any bounded positive step function alternative density  $f_1$  on  $(0,1)$ ,

$$n^{1/2} \left( \frac{n}{n+2-2m} \right)^{1/2} [H_{m,n} + \log(2m) + \gamma - R(1, 2m-1) - H(f_1)]$$

is asymptotically normal with mean zero and variance  $\tau(2m) + \text{Var}_{f_1} \log f_1$ , where

$$\tau(k) = \begin{cases} \pi^2/6 - 1 & \text{if } k = 1 \\ (2k^2 - 2k + 1) \left\{ \frac{\pi^2}{6} - \left( 1 + \frac{1}{2^2} + \dots + \frac{1}{(k-1)^2} \right) \right\} - 2k + 1 & \text{if } k \geq 2. \end{cases}$$

Hence an asymptotic approximation to the power of a level  $\alpha$  test under alternative  $f_1$  is

$$\Phi \left( \frac{n^{1/2} \left( \frac{n}{n+2-2m} \right)^{1/2} [H_{\alpha}^*(m,n) + \log(2m) + \gamma - R(1,2m-1) - H(f_1)]}{\sqrt{\tau(2m) + \text{Var}_{f_1} \log f_1}} \right)$$

(1.13)

## 2. ONE-SAMPLE EVALUATION BASED ON ASYMPTOTIC PERCENTILE

### RANKS

We now use test (1.7) to evaluate whether each of nine of the uniform random number generators available in Dudewicz and Ralley (1981) have entropies consistent with uniformity. At the same time, the form of the test (1.7) suggests that if, for two random number generators  $\alpha_1$  and  $\alpha_2$ ,

$$H_{m,n}(\alpha_1) < H_{m,n}(\alpha_2),$$

then  $\alpha_2$  is "better than"  $\alpha_1$ . This provides a basis for comparative evaluation of the random number generators. Thus we first obtain  $H_{m,n}(\alpha_i)$  for each of nine of the random number generators  $\alpha_i$  in Dudewicz and Ralley (1981): KNCG (using IX = 1, L1 = 452807053, C = 0.0, PIP = 2.\*\*31), SRAND, KERAND, UNI, RN1, RN2, RN3, RN4 and RN5.

Fixing  $n = 10,000$  and letting  $m = 1(1)10(5)20(10)40$ , comparison between the results obtained by Monte Carlo methods and by asymptotic methods is one of our goals. This value of  $n$  automatically excluded RN5 from consideration for  $m < 5$  since the period of RN5 is only 2048 ( $=2^{11}$ ), hence leading to definite rejection due to gaps of size zero for small  $m$ .

For each of the 9 generators, the statistics  $H_{m,10000}$  were computed from a single sample of 10,000

numbers for 14 values of  $m$  (10 for RN5) as above. The percentile ranks of these statistics, expressed in terms of their asymptotic distributions under  $H_0$  (see equation (1.10)), were plotted in Figure 1.

Figure 1

Examination of that graph reveals that, among all the generators considered, RN2, RN4, and KERAND have entropy estimates most consistent, almost too consistent, with uniformity, while UNI gives good entropy estimates in the same sense. The poorest generator is RNCG, while the rest performed reasonably well. (To be rejected, the  $H_{m,n}$  - value of a generator needs to have a percentile rank  $\leq 5.00$ , at the top of the graph scale.) These interpretations are based on a single sample of size 10,000 and therefore should be regarded more as an illustration of use of  $H_{m,n}$  for comparing random number generators than as a definitive statement regarding them. More replications are used below in a way reminiscent of the extensive testing of random number generators reported in Dudewicz and Ralley (1981).

Since the critical region for the entropy test involves the lower (left) tail of the null distribution of  $H_{m,n}$ , the (asymptotic) cutoff point, say for  $\alpha = .05$ , corresponds to the 5% point of the asymptotic null

distribution and is seen to be to the left of most points in Figure 1. Thus, it is clear from the plots in Figure 1 that all generators under study except RNCG pass this test of uniformity for all the values of  $m$  considered. (Of course RN5 fails for  $m < 5$ .)

### 3. CONVERGENCE OF $H_{m,n}$ TO ASYMPTOTIC NULL DISTRIBUTIONS

---

Dudewicz and van der Meulen (1979) reported Monte Carlo percentage points of the null distribution of  $H_{m,n}$  for 10,000 replications of sizes  $n = 10(10)50(50)100$  each using the generator UNI (using ISEED = 524287 and JSEED = 654345465). A pilot study showed that extending this to 10,000 replications of size 10,000 each would be feasible but expensive.

In the same report, the above authors also studied the normal approximation to the percentage point  $H_{\alpha}^{*}(m,n)$  given by (1.11) for  $\alpha = .05$ , and a variety of  $m,n$  up to  $n \leq 100$ . It was found that the asymptotic cutoffs deviate by a large amount from the simulated cutoff points, at least for  $n \leq 100$ .

In this report we continue the investigations of the above authors and investigate the normal approximation to percentage points for  $n = 10,000$ . For this we first obtain estimates of the true percentage points based on a Monte Carlo study using 1,000 samples of size 10,000. The resulting Monte Carlo estimates are given in Table 1, for various levels of  $\alpha$  ( $\alpha = .40, .30, .20, .10, .05, .01$ ) and  $m = 1(1)10(5)20(10)40$ . We also obtained the corresponding asymptotic values of  $H_{\alpha}^{*}(m,n)$ , for  $n = 10,000$  and the same values of  $\alpha$  and  $m$ . These asymptotic values are listed in

Table 1, together with the Monte Carlo estimates, and it is seen that - for such a large sample size as  $n = 10,000$  - the asymptotic cutoffs are - for  $m$  up to 10 - in close agreement with the simulated ones using UNI, and based on 1000 replications.

Table 1
---------

In addition to the above we plotted the asymptotic distributions of the  $H_{m,n}$  - statistics for  $n = 10,000$  and  $m = 1(1)10(5)20(10)50, 250$  in Figures 2a-2p. This asymptotic distribution is, for  $n \rightarrow \infty$ ,  $m = o(n^{1/3-\delta})$ , and  $\delta > 0$ , given in an explicit form by (cf. 1.10)

$$(-\log(2m) - \gamma + R(1, 2m-1), \frac{1}{6mn}), \quad (3.1)$$

where  $(\mu, \sigma^2)$  denotes a normal distribution with mean  $\mu$  and variance  $\sigma^2$ . In each figure the asymptotic distribution function of  $H_{m,n}$  is overlaid with the corresponding simulated percentage points of  $H_{m,n}$  obtained from the Monte Carlo study described above for  $\alpha = .005, .010, .025, .050, .100, .200, .300, .400$ . It is seen from these figures that the normal approximation to the simulated values is good for all  $m$ -values up to 4, bad beyond 10, and questionable in the range inbetween. The goodness depends on the use to which the approximation is to be put. For the use in Section 5 below it is bad in the

range  $m > 4$ . (From the restriction on  $m$  one expects goodness at best for  $m \leq n^{1/3} \approx 22$ .)

FIGURES  
2a-2p

The fact that the normal approximation to the distribution of  $H_{m,n}$  deteriorates for  $n = 10,000$  as  $m > 10$ , as observed from Figures 2a-2p, also has a bearing on the study reported in Section 2. There, for one sample of size  $n = 10,000$  the percentile ranks of the observed values of  $H_{m,n}$  with respect to the asymptotic distributions are plotted for nine random number generators as functions of  $m$ . Strictly speaking, this approximation is not justified for values of  $m > n^{1/3} \approx 22$  on the basis of Theorem 1.8. The figures 2a-2p indicate that the asymptotic distribution moves more and more to the right of the percentile points obtained through simulation as  $m$  increases,  $m \geq 15$ , i.e. the asymptotic distribution is more and more significantly stochastically larger than the actual (simulated) distribution. From this it can be surmised that for  $m \geq 15$  the actual percentile ranks of the observed  $H_{m,n}$  - values in the study leading to Figure 1 are larger than the ones plotted there, which are based on the asymptotic distribution, and that the discrepancies get larger as  $m$  increases. This may explain the general upward trend in Figure 1 of the graphs



as  $m$  increases beyond 15; the actual percentile ranks are larger than the plotted ones for  $m \geq 15$ , which, when corrected for, would yield a more horizontal behavior of the percentile rank curves for  $m \geq 15$  for the random number generators under consideration.

#### 4. CONVERGENCE OF $H_{m,n}$ TO ASYMPTOTIC ALTERNATIVE

##### DISTRIBUTIONS

Expression (1.13) for the asymptotic power of the test against alternatives which are bounded positive step functions on  $[0,1]$  was used by Dudewicz and van der Meulen (1979) to compute an approximation to the power for nine alternative distributions for  $n = 20$ ,  $m = 1, 2, 4, 9$  and  $\alpha = .05$ . Some of these alternative distributions belong to the class  $\mathcal{F}$  of bounded positive step functions, others do not. As a general conclusion, the above authors found that the asymptotic power approximation agreed poorly with the Monte Carlo power evaluations for the values of  $n$ ,  $m$ , and  $\alpha$  considered. These investigations were continued for  $n = 100$  and it was found that the normal approximation is still not accurate.

In this paper we extend the evaluation of the normal approximation to power for a particular alternative (called  $F$  in Dudewicz and van der Meulen (1979)) for  $n = 100$ ,  $1000$ , and  $10,000$  and  $\alpha = .05$ . This alternative which, belongs to  $\mathcal{F}$ , is defined by the following distribution function:

$$\begin{aligned}
F: F(x) &= x && \text{if } 0 \leq x \leq 1/4 - \epsilon/2, \\
F(x) &= \delta(1/4 - \epsilon/2) + x(1 - \delta), && \text{if } 1/4 - \epsilon/2 \leq x \leq 1/4 + \epsilon/2, \\
F(x) &= x - \epsilon\delta, && \text{if } 1/4 + \epsilon/2 \leq x \leq 3/4 - \epsilon/2, \\
F(x) &= -\delta(3/4 + \epsilon/2) + x(1 + \delta), && \text{if } 3/4 - \epsilon/2 \leq x \leq 3/4 + \epsilon/2, \\
F(x) &= x, && \text{if } 3/4 + \epsilon/2 \leq x \leq 1,
\end{aligned}$$

(with  $0 < \epsilon < 0.5$ ,  $0 < \delta \leq 1$ ; we used  $\epsilon = .02$ ,  $\delta = 0.1$ ).

In order to carry out the asymptotic power evaluation for alternative F we first need the cutoff points  $H_{\alpha}^*(m, n)$  of our test procedure (1.7) for various values of  $\alpha$ ,  $m$ ,  $n$ , and, moreover, Monte Carlo estimates of the power to compare the asymptotic approximation with. Monte Carlo estimates of  $H_{\alpha}^*(m, n)$  for  $n = 100$  and various values of  $\alpha$  and  $m$  are given in Dudewicz and van der Meulen (1979). Monte Carlo estimates of  $H_{\alpha}^*(m, n)$  for  $n = 1000$  (based on  $N = 10,000$  replications, and for  $m = 1(1)10(5)20(10)50, 250$ ) and for  $n = 10,000$  (based on  $N = 1000$  replications, and for  $m = 1(1)10(5)20(10)50, 250, 500, 2500$ ) are given in Table 2 and Table 3 respectively for  $\alpha = .005, .01, .025, .05, .10, .20, .30, .40$ .

Table 2
---------

Table 3
---------

Monte Carlo estimates for the power of the entropy-based test against alternative F for  $n = 100, 1000$ , and

10,000 (based on 10,000, 10,000, and 1000 replications respectively) are given in Table 4 for  $\alpha = .05$  and  $m = 1(1)10(5)20(10)40$ , together with the asymptotic power calculated in each case using formula (1.13). Both the Monte Carlo estimates ("simulated power") and the asymptotic power are obtained using the Monte Carlo estimates of  $H_{\alpha}^*(m,n)$  ("simulated cutoffs") described above.

In calculating the asymptotic power note that - for the density  $f_1$  of  $F - H(f_1) = -.000002$ ,  $\text{Var } f_1 \log f_1(X) = -.000003842$ , and  $\gamma = .5772...$

It is seen from Table 4 that there is reasonable agreement for  $n = 10,000$  (and  $1 \leq m \leq 3$ ), but for  $n = 100$  and  $n = 1,000$  there seems to be poor agreement overall, except for  $m = 1$  at  $n = 1,000$ .

Table 4
---------

## 5. EVALUATION OF NINE RANDOM NUMBER GENERATORS BASED ON A $\chi^2$ -TEST

In Section 2 we presented a one-sample evaluation of nine random number generators based on the percentile ranks of the observed  $H_{m,n}$ -values within the asymptotic distribution.

As a more sensitive approach we ran a  $\chi^2$ -test using  $N = 100$  sample-values of  $H_{m,n}$  ( $n = 10,000$ ;  $m = 1(1)4$ ) from the same nine (without RN5) random number generators to determine whether the distributions of  $H_{m,n}$  fit the corresponding asymptotic distributions which, for  $m \leq 4$ , have been verified in Section 3 as being very close to the true distributions. The results are given in Table 5.

Table 5
---------

For carrying out the  $\chi^2$ -test for goodness of fit we used 10 equi-probable cells. The null-hypothesis entertained in each case is that the 100 sample values of  $H_{m,n}$  come from the asymptotic normal distribution given by (3.1). The entries in Table 5 are the observed  $\chi^2$ -values for the eight random number generators and four values of  $m$ . Those  $\chi^2$ -values which are significant at the .05 level are starred.

From the observed  $\chi^2$ -values we conclude that seven random number generators produce numbers which are consistent with the hypothesis of randomness as measured by the Chi-square entropy-test, whereas one random number generator (RN3) is to be rejected due to the lack of proper distribution of the  $H_{m,n}$  - statistic on the basis of samples of size  $n = 10,000$ .

## 6. CONCLUSIONS

In this paper we looked at the performance of nine random number generators in terms of the entropy-based test statistic  $H_{m,n}$ . A one-sample evaluation of size  $n = 10,000$  indicated that all (except two) random number generators easily pass the entropy-based test, but it also showed that among the various generators some are closer to the uniform distribution than others as measured by the value of  $H_{m,n}$ . One random number generator was excluded from further study since its small period leads to rejection at every level.

The study of the convergence of  $H_{m,n}$  to the asymptotic null distribution (initiated in Dudewicz and van der Meulen (1979) for sample sizes up to  $n = 100$ , and continued here for  $n = 10,000$ ) revealed that for  $n = 10,000$  and  $m \leq 10$  the normal approximation to the percentage points (and thus to the distribution of  $H_{m,n}$ ) is good. This comparison was carried out in two ways: i) by a comparison of simulated cutoff points with those found through the asymptotic formula (1.11), and (ii) by a plot of the asymptotic normal distribution of  $H_{m,n}$  with the

simulated values drawn into it. The latter study confirmed the first one.

Next we investigated the convergence of  $H_{m,n}$  to the asymptotic alternative distribution under a specific alternative  $F$  which, because it is so close to uniform, is hard to detect. We observed that the simulated power of the entropy-based test is small for  $n = 10,000$  but for some  $m$ 's definitely larger than the significance level. We noticed that the asymptotic approximation to power for  $n = 10,000$  as given by formula (1.13) has validity only for  $m \leq 3$ . In the course of these investigations we obtained Monte Carlo estimates of the percentage points for  $n = 1,000$  and  $n = 10,000$  which supplement the tables already provided in Dudewicz and van der Meulen (1979).

Finally we evaluated the eight remaining random number generators on basis of a  $\chi^2$ -test. The purpose of this test was to investigate whether the true distribution of  $H_{m,n}$  fitted the asymptotic null distributions closely, which we know from the results in Section 3 they should for  $m = 1, 2, 3, 4$  if the generators are truly random. On basis of this study the random number generator RN3 was rejected.



#### REFERENCES

- [1] Dudewicz, E. J. and van der Meulen, E. C. (1979):  
"Entropy-based statistical inference, I:  
Testing hypotheses on continuous probability  
densities with special reference to uniformity,"  
Mededelingen uit het Wiskundig Instituut,  
Katholieke Universiteit Leuven N°120, Belgium.
- [2] Dudewicz, E. J. and Ralley, T. G. (1981): The  
Handbook of Random Number Generation and Testing  
with TESTRAND Computer Code, American Sciences  
Press, Inc., Columbus, Ohio.
- [3] Vasicek, O. (1976): "A test of normality based on  
sample entropy," J.R. Statist. Soc. B, 38,  
54-59.



Figures 2a-2p. The asymptotic distribution function of  $H_{m,n}$  for  $n=10,000$  and  $m=1(1)10(5)20(10)50,250$ ; \* represents a Monte Carlo value at  $\alpha=.005,.01,.025,.05,.10,.20,.30,.40$ .

Figure 2a.

$N= 10000$

$M= 1$

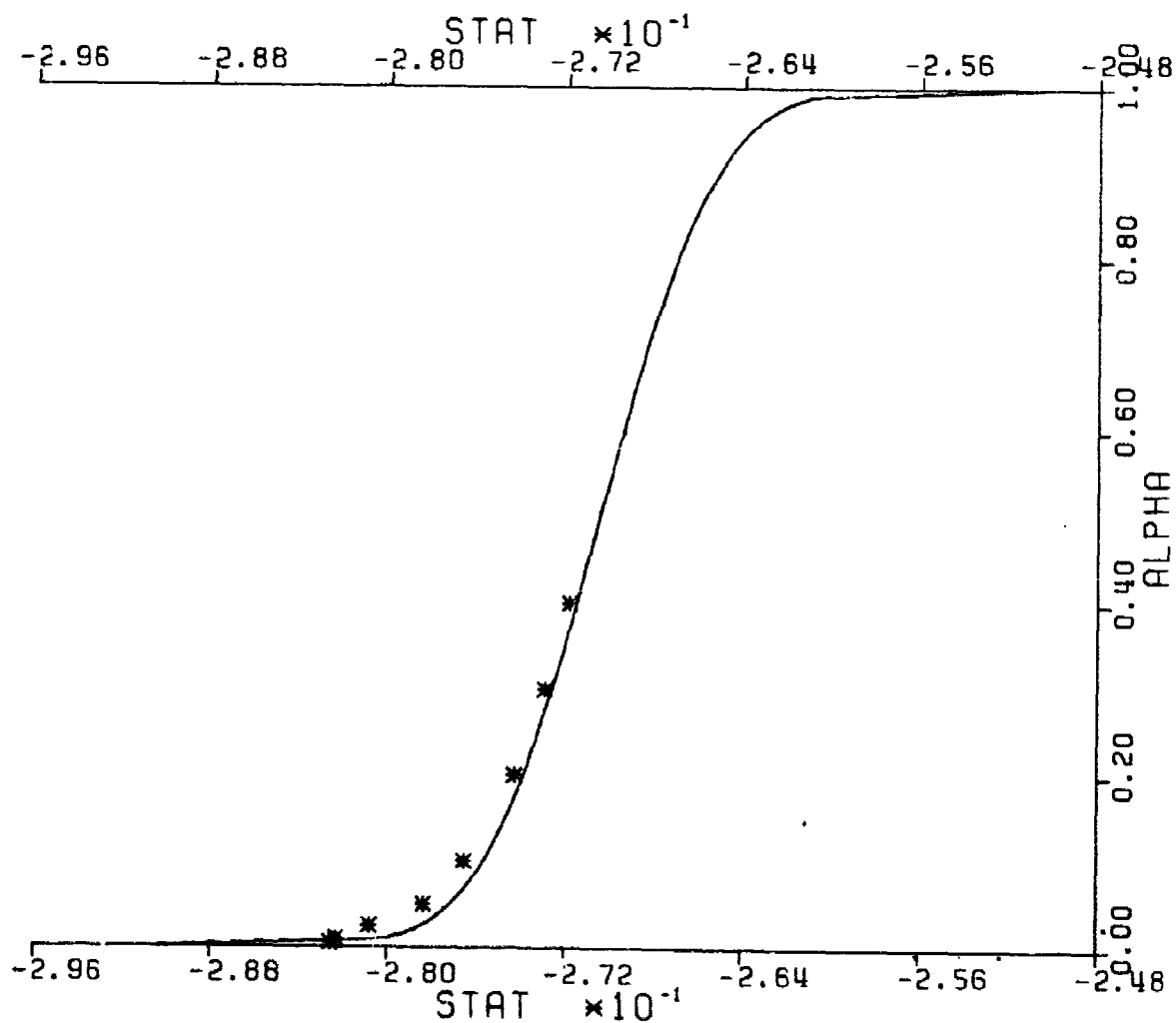


Figure 2b.

N= 10000

M= 2

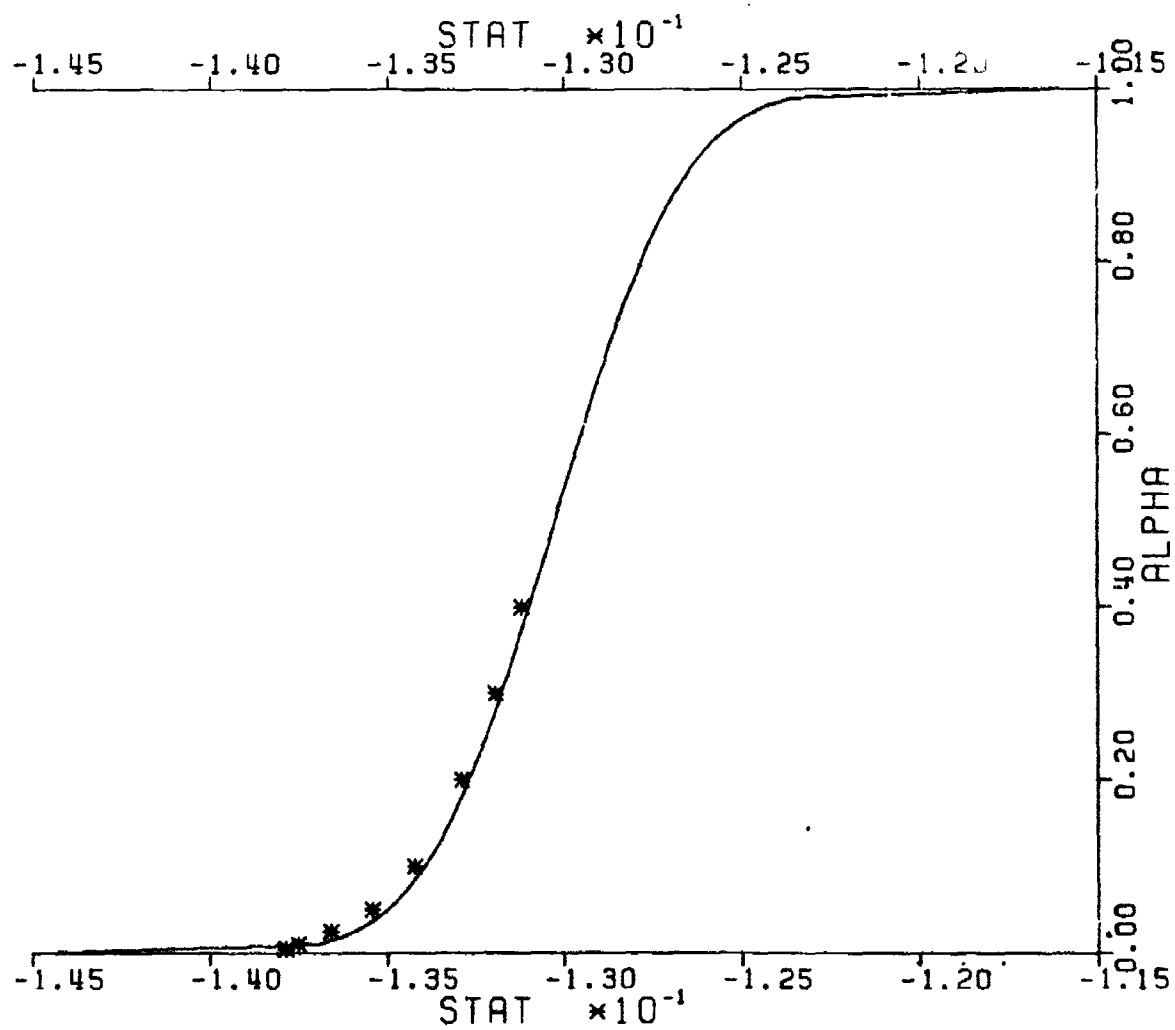


Figure 2c.

N= 10000

M= 3

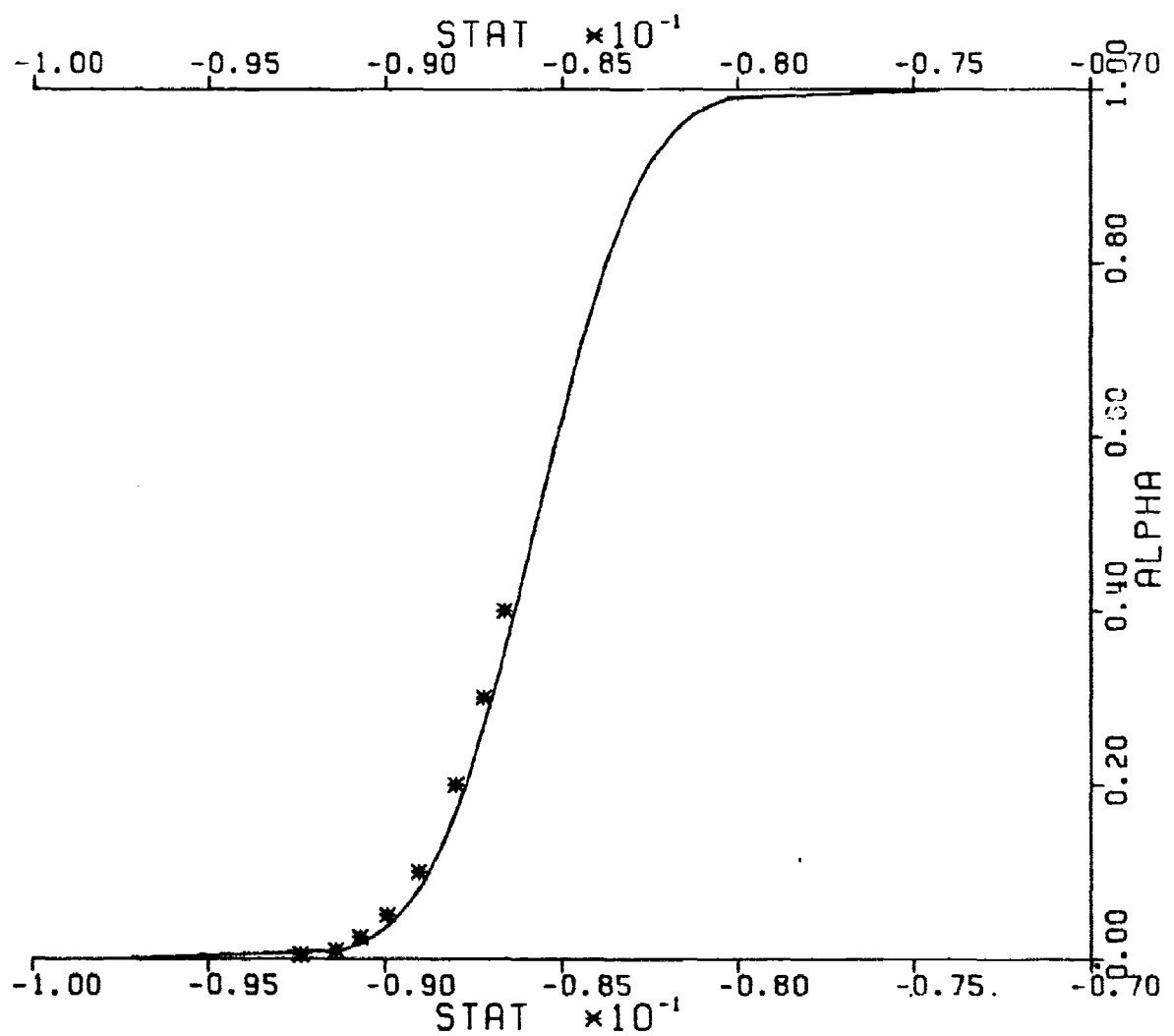


Figure 2d.

N= 10000

M= 4

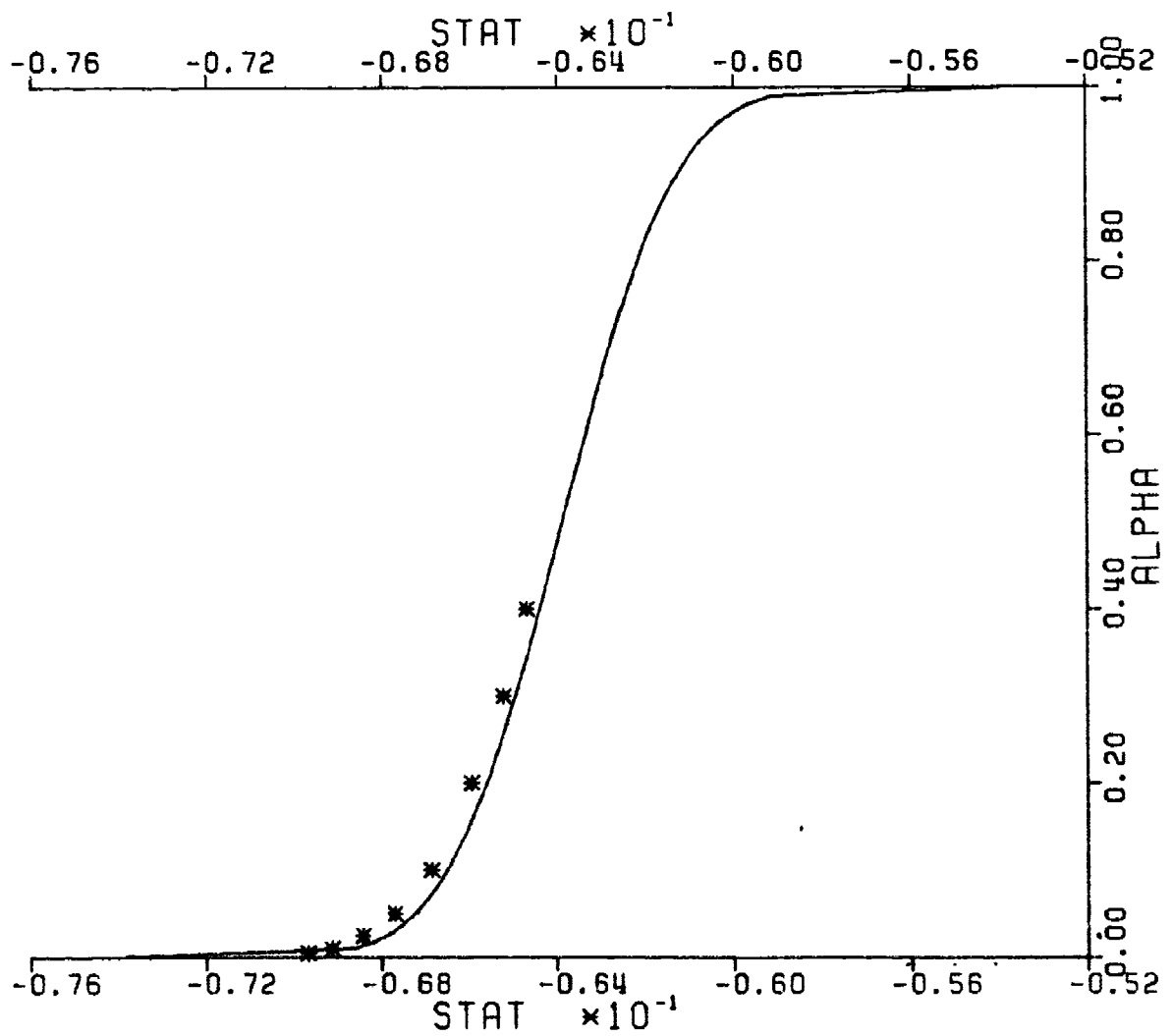


Figure 2e.

N= 10000

M= 5

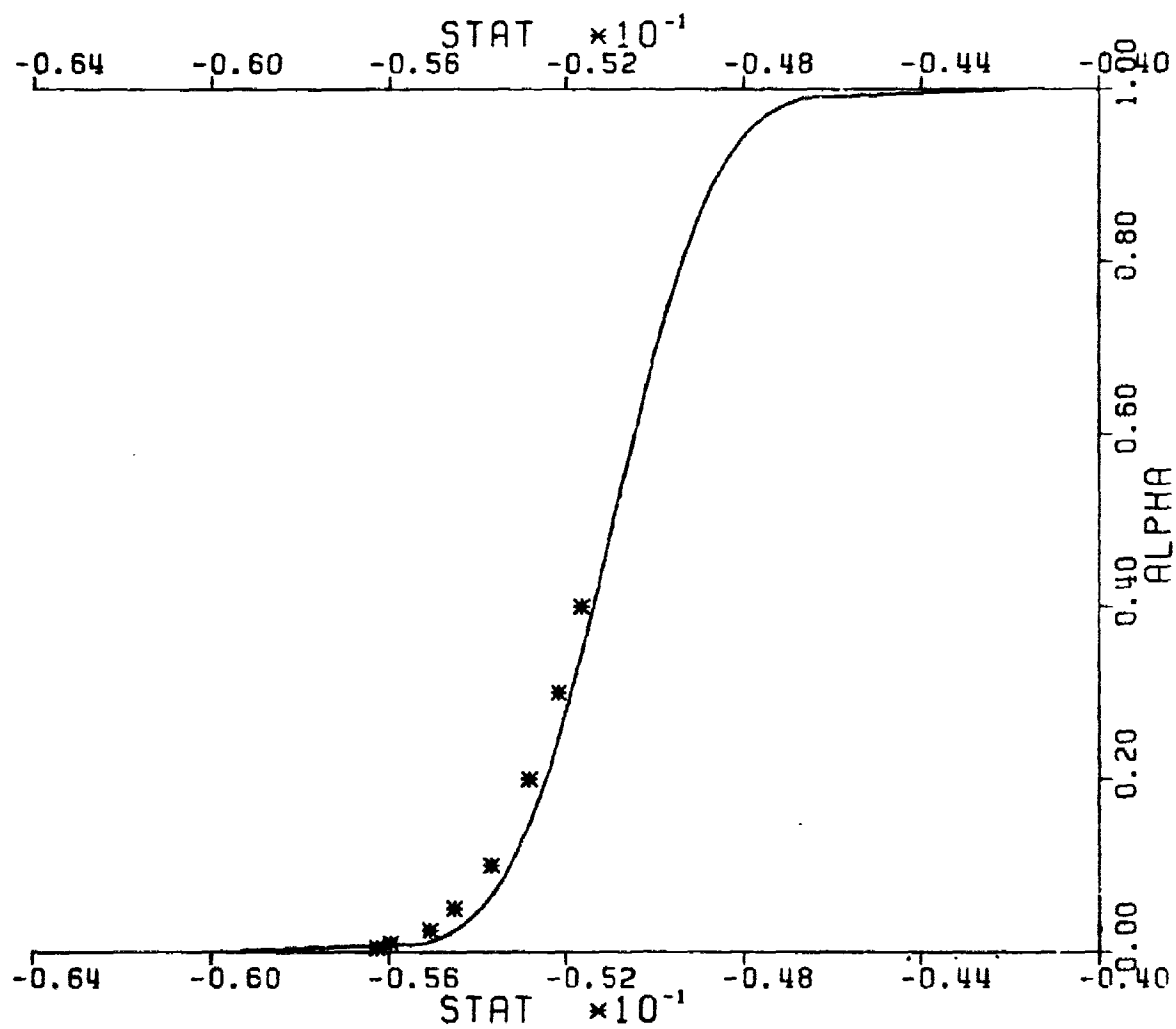


Figure 2f.

N= 10000

M= 6

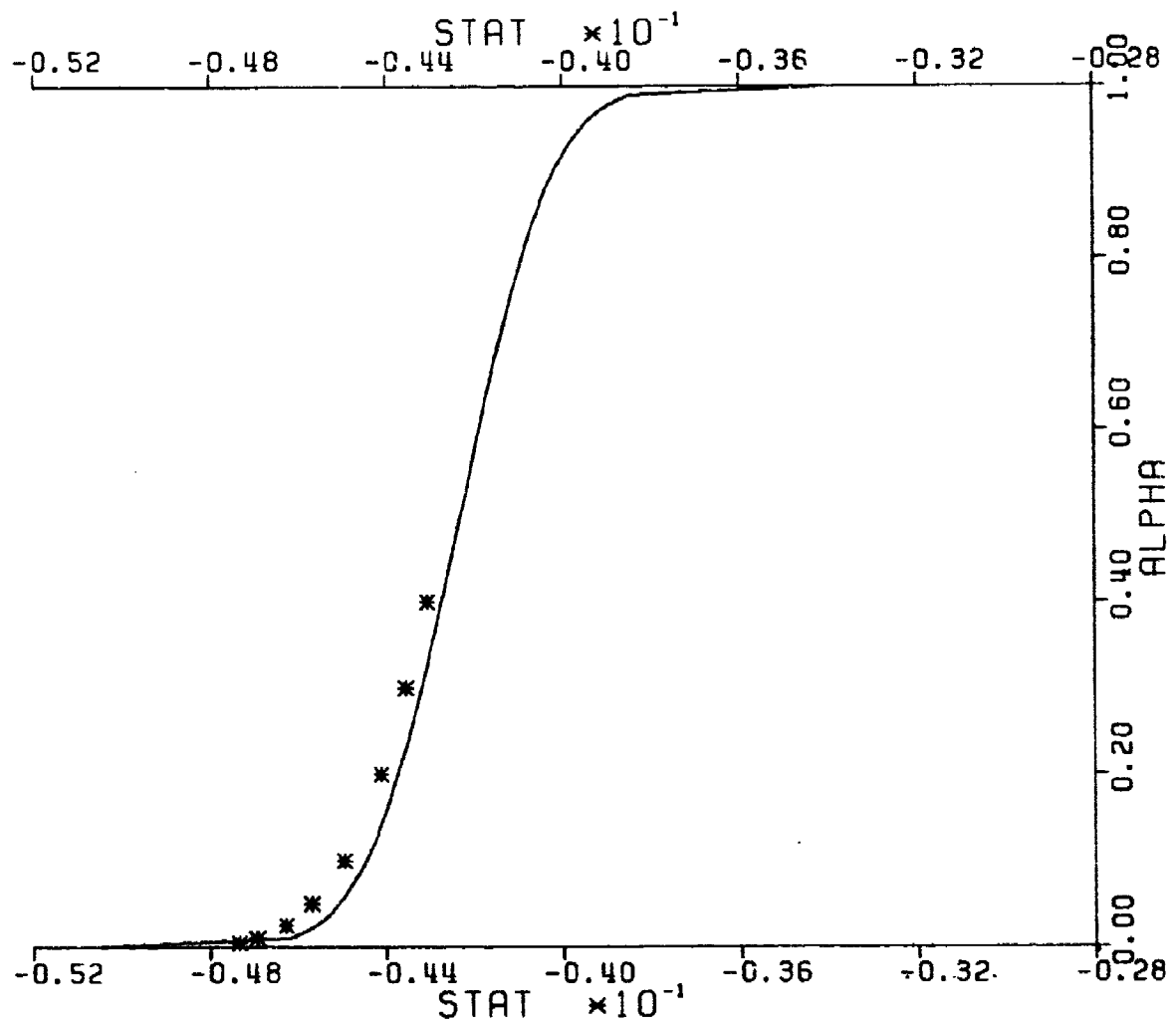




Figure 2g.

N= 10000

M= 7

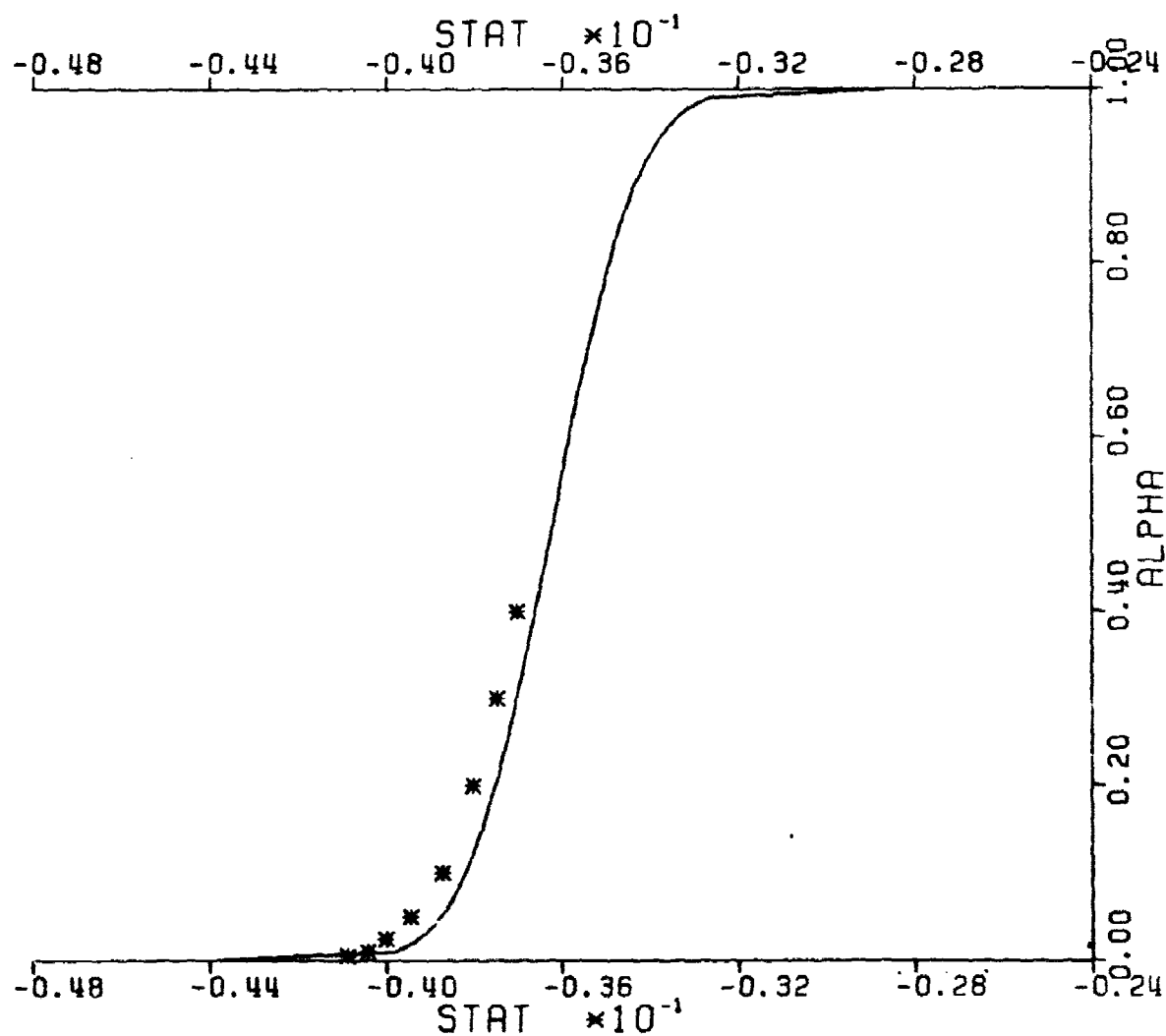


Figure 2h.

N= 10000

M= 8

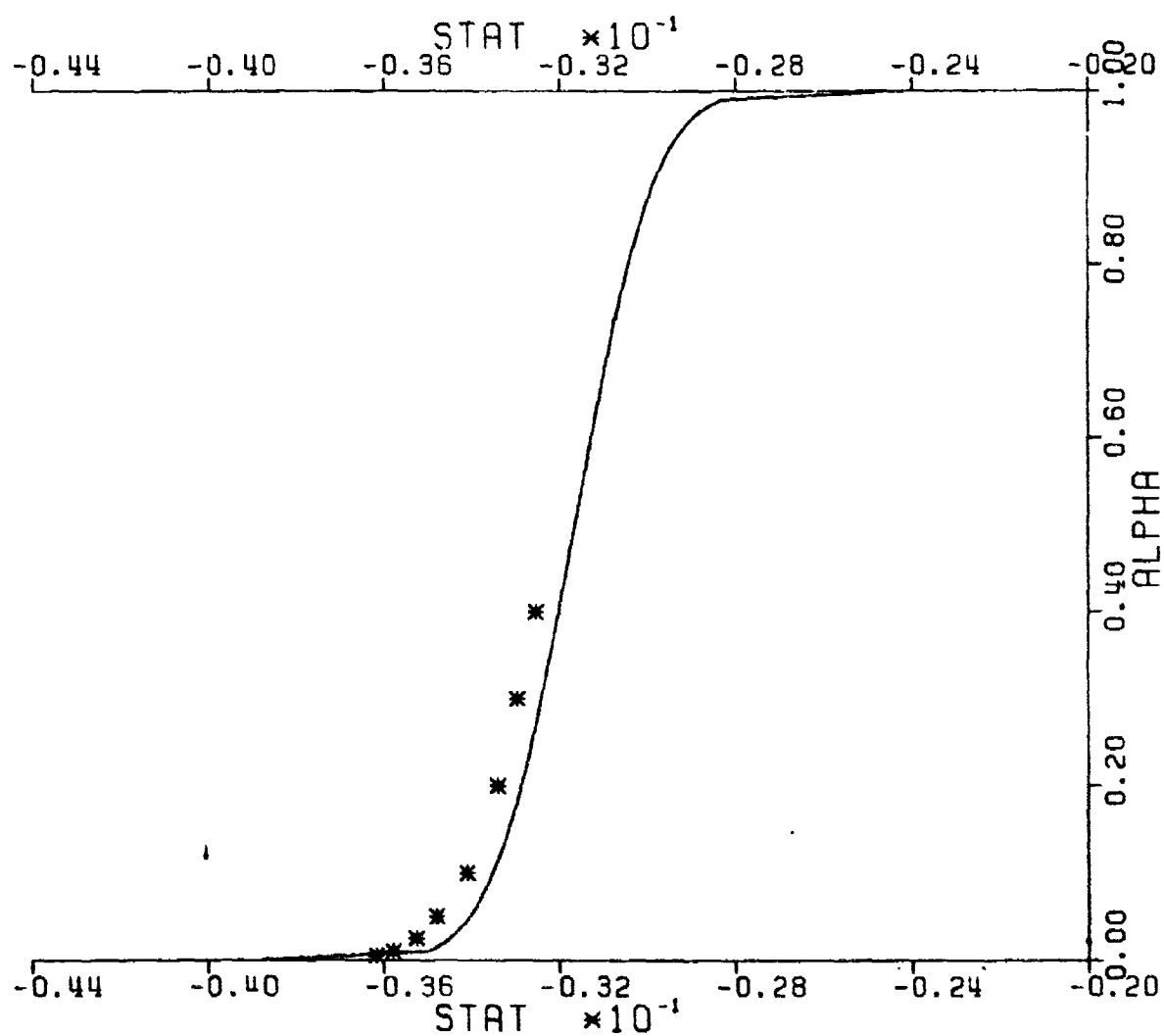


Figure 2i.

N= 10000

M= 9

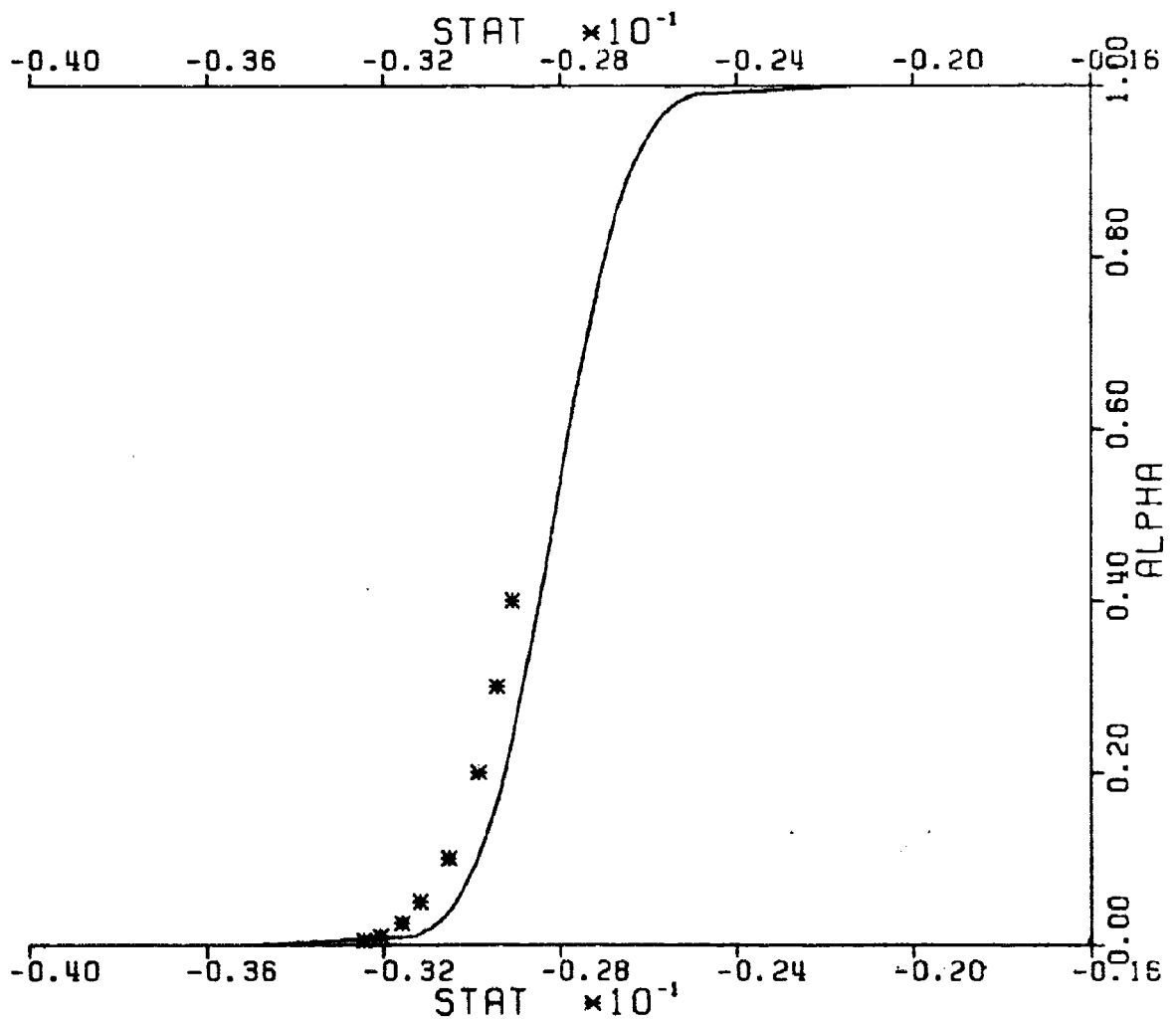


Figure 2j.

N= 10000

M= 10

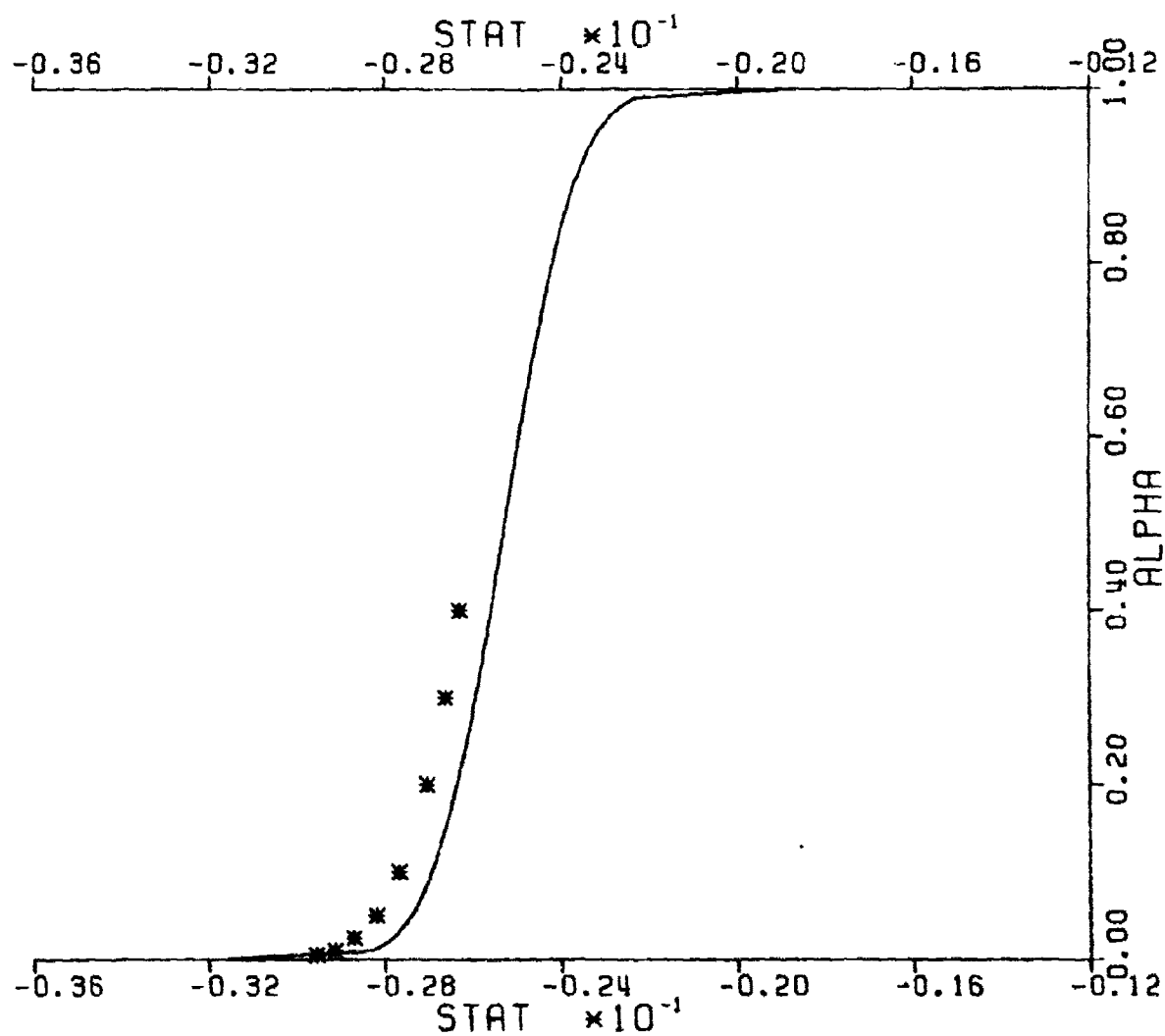


Figure 2k.

N= 10000

M= 15

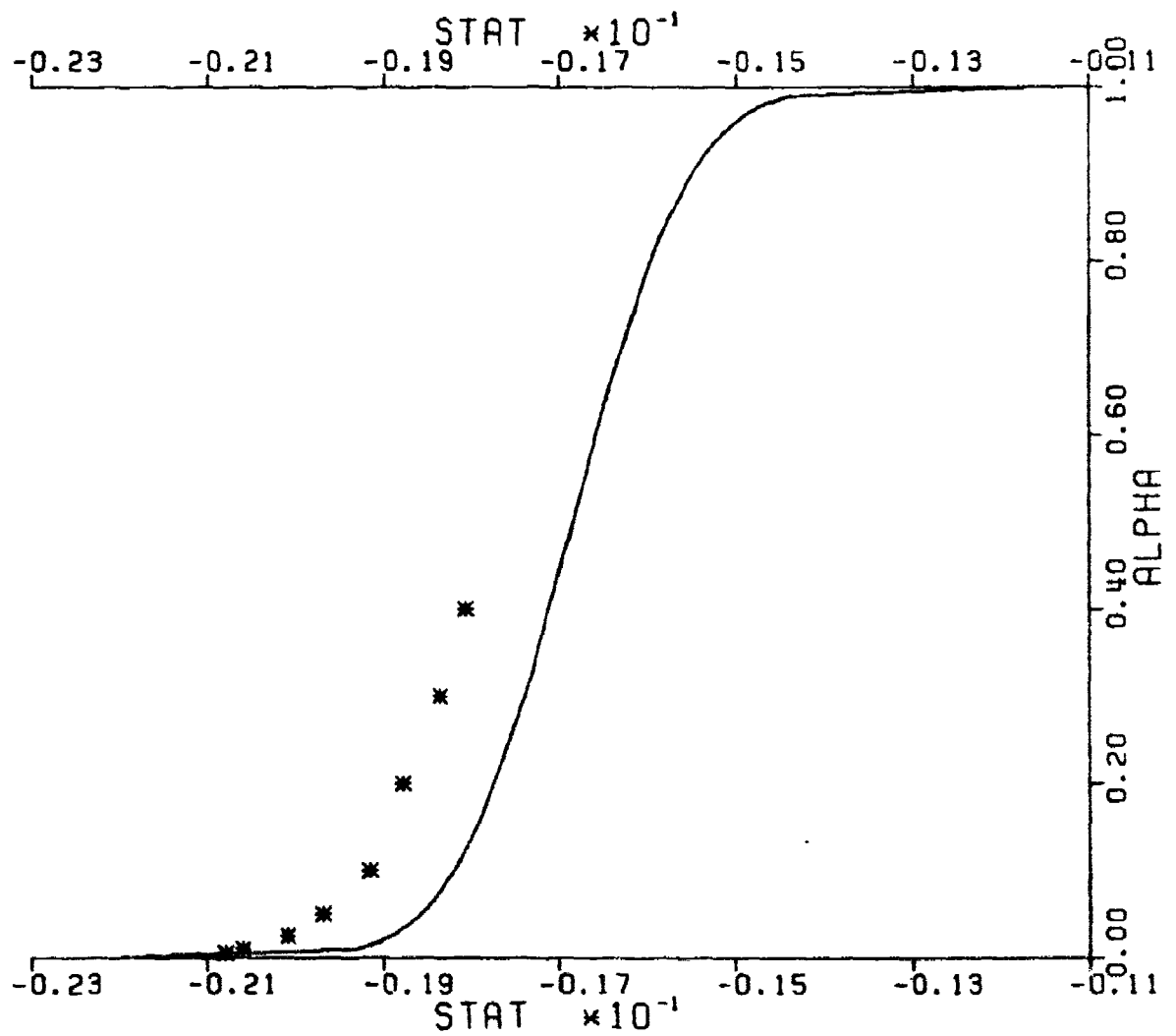


Figure 21.

N= 10000

M= 20

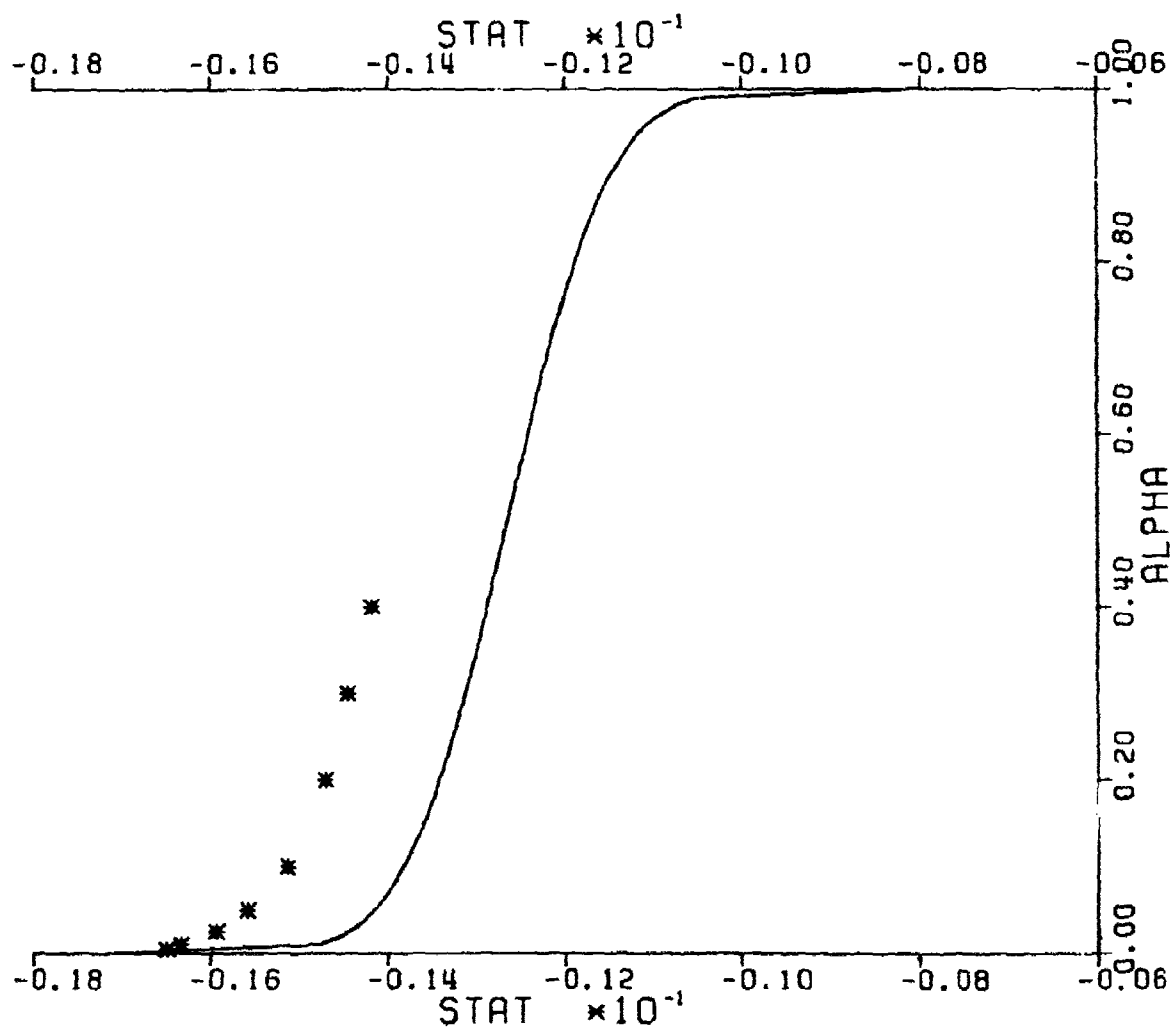


Figure 2m.

N= 10000

M= 30

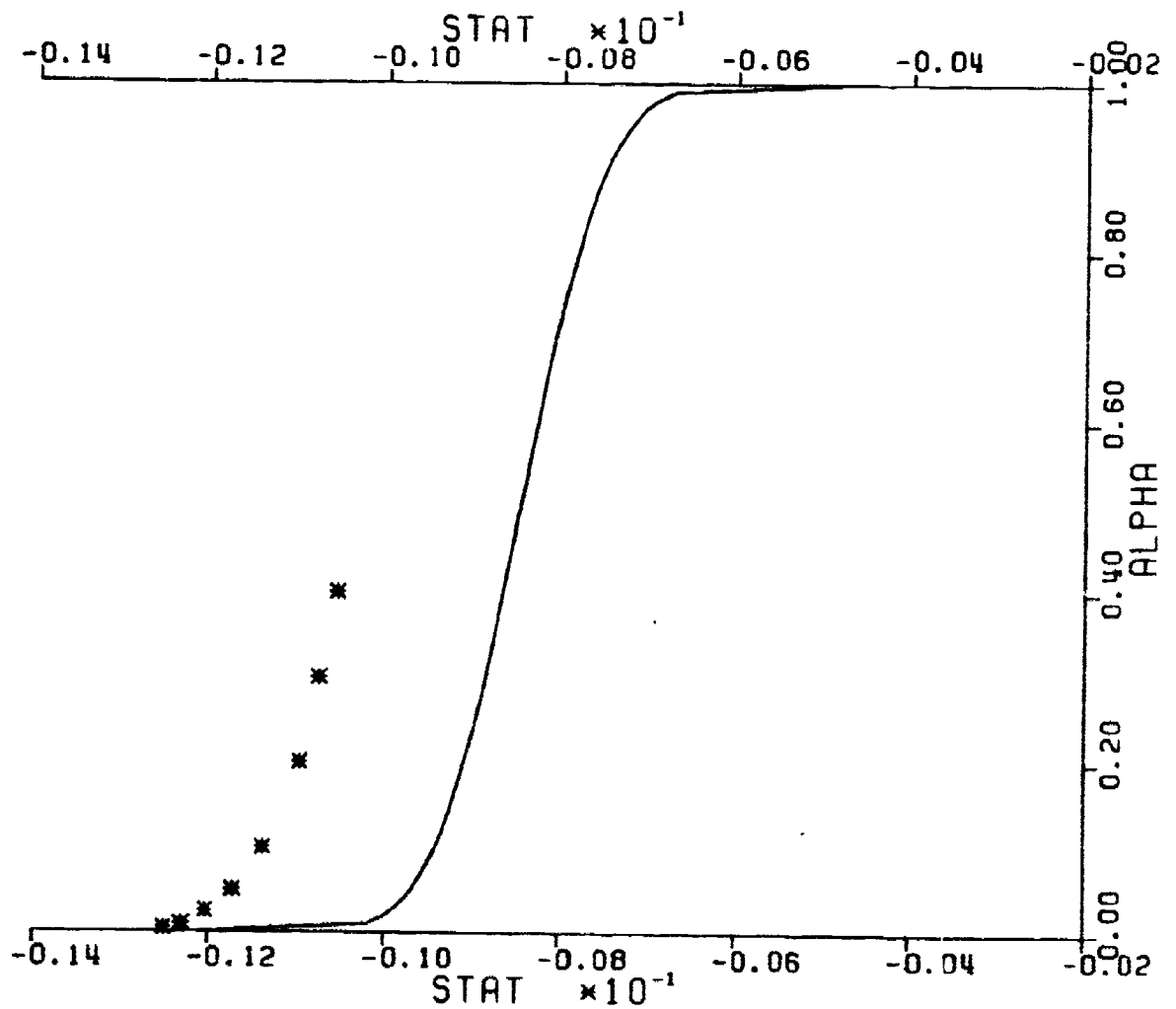


Figure 2n.

N= 10000

M= 40

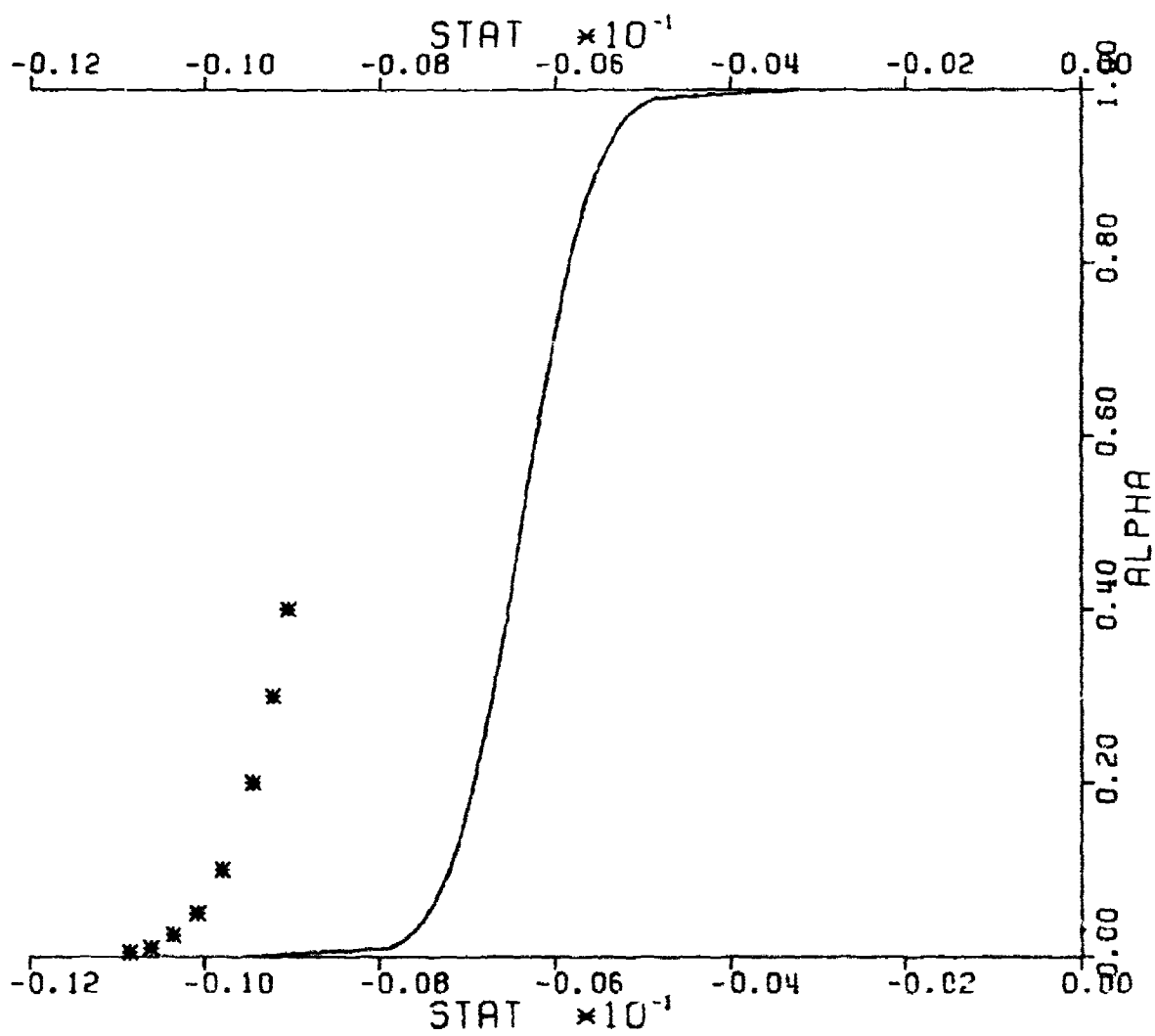




Figure 2o.

N= 10000

M= 50

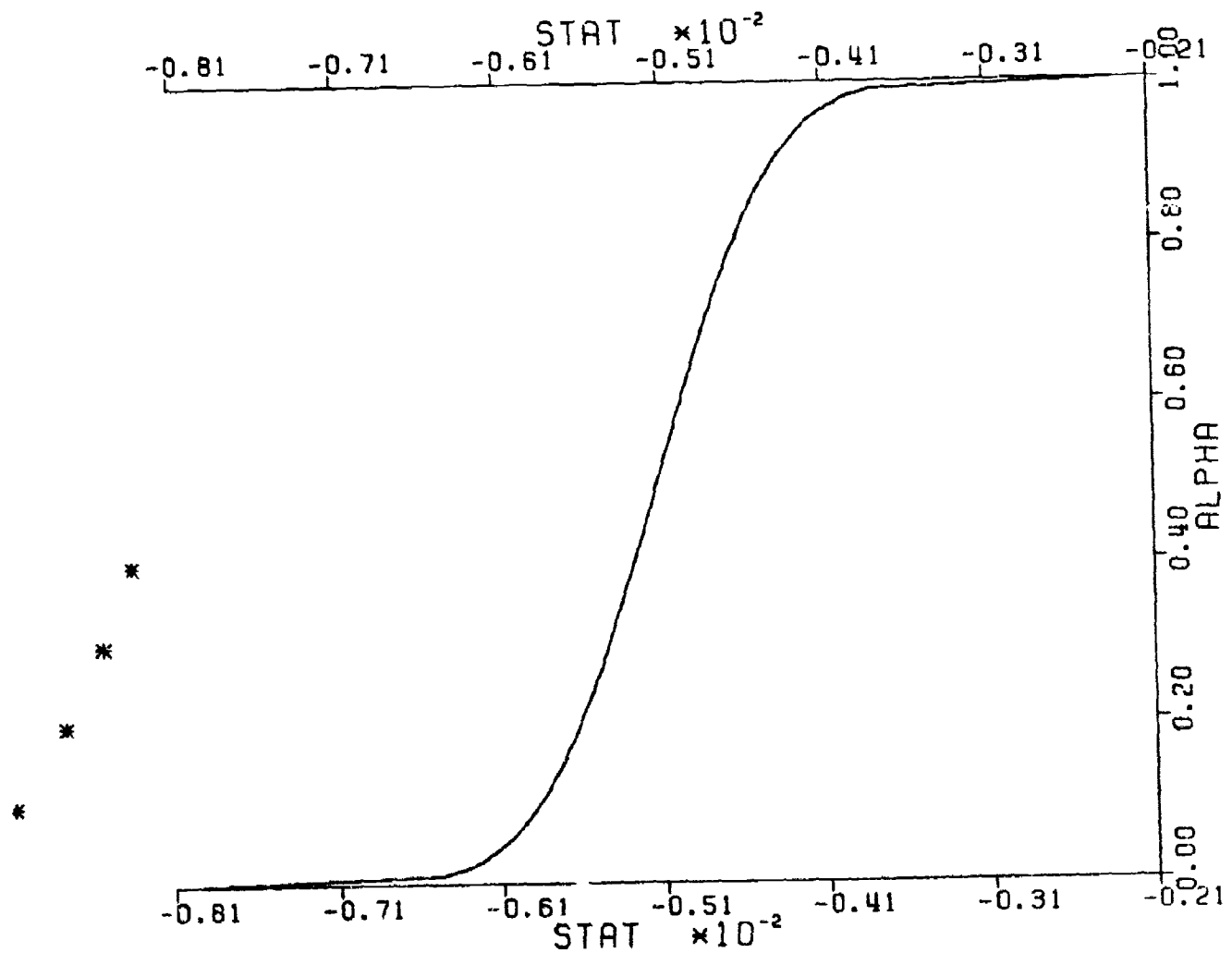


Figure 2p.

N= 10000

M= 250

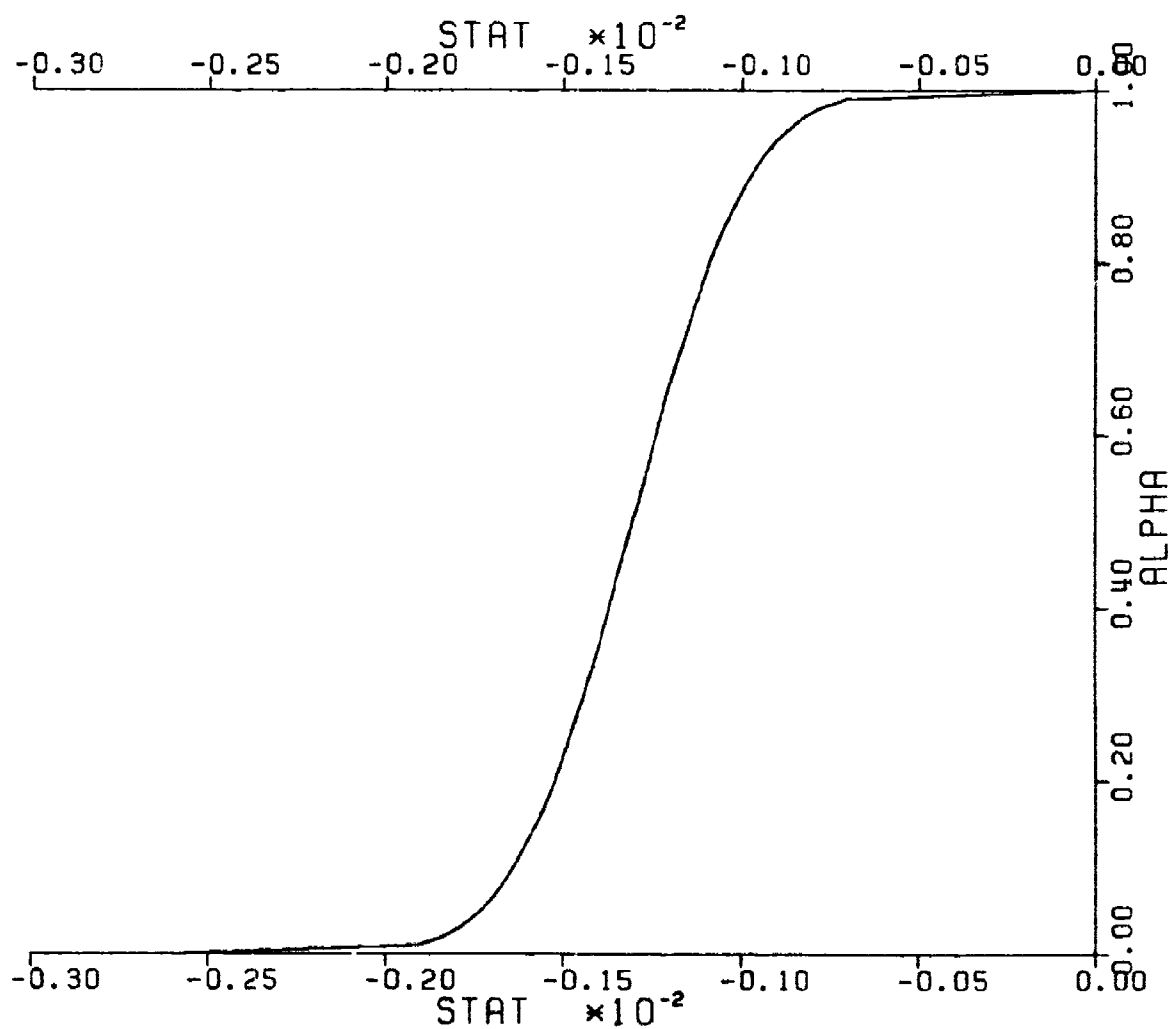


TABLE 1

COMPARISON OF MONTE CARLO (1000 REPLICATIONS, N=10000)

AND

ASYMPTOTIC CUTOFFS OF H(N,N), N=10000

H	ALPHA	H(N,N)	A-H(N,N)	ALPHA	H(N,N)	A-H(N,N)	ALPHA	H(N,N)	A-H(N,N)
1	0.4000	-0.271700	-0.271479	0.2000	-0.274281	-0.273881	0.0500	-0.278373	-0.277162
2	0.4000	-0.131208	-0.130990	0.2000	-0.132897	-0.132689	0.0500	-0.135420	-0.135009
3	0.4000	-0.084597	-0.084322	0.2000	-0.087994	-0.087708	0.0500	-0.089921	-0.089603
4	0.4000	-0.064700	-0.064491	0.2000	-0.065933	-0.065602	0.0500	-0.067693	-0.067243
5	0.4000	-0.051635	-0.051379	0.2000	-0.052815	-0.052453	0.0500	-0.054529	-0.053921
6	0.4000	-0.043078	-0.042731	0.2000	-0.044126	-0.043733	0.0500	-0.045701	-0.045073
7	0.4000	-0.037020	-0.036616	0.2000	-0.038011	-0.037524	0.0500	-0.039466	-0.038764
8	0.4000	-0.032345	-0.032027	0.2000	-0.033393	-0.032877	0.0500	-0.034794	-0.034037
9	0.4000	-0.029083	-0.028466	0.2000	-0.029844	-0.029267	0.0500	-0.031164	-0.030361
10	0.4000	-0.026313	-0.025523	0.2000	-0.027061	-0.026383	0.0500	-0.028201	-0.027421
15	0.4000	-0.018074	-0.017118	0.2000	-0.018766	-0.017739	0.0500	-0.019689	-0.018586
20	0.4000	-0.014176	-0.012878	0.2000	-0.014703	-0.013415	0.0500	-0.015572	-0.014149
30	0.4000	-0.010338	-0.008647	0.2000	-0.010956	-0.009086	0.0500	-0.011720	-0.009685
40	0.4000	-0.009040	-0.006837	0.2000	-0.009433	-0.006917	0.0500	-0.010072	-0.007436
1	0.3000	-0.272059	-0.272385	0.1000	-0.276558	-0.275679	0.0100	-0.282290	-0.279945
2	0.3000	-0.131953	-0.131773	0.1000	-0.134215	-0.133960	0.0100	-0.137520	-0.136977
3	0.3000	-0.087202	-0.086961	0.1000	-0.089042	-0.088746	0.0100	-0.091378	-0.091209
4	0.3000	-0.065211	-0.064934	0.1000	-0.066069	-0.066501	0.0100	-0.069138	-0.068834
5	0.3000	-0.052163	-0.051874	0.1000	-0.053676	-0.053257	0.0100	-0.055964	-0.055165
6	0.3000	-0.043560	-0.043203	0.1000	-0.044932	-0.044466	0.0100	-0.046938	-0.046208
7	0.3000	-0.037487	-0.037034	0.1000	-0.038720	-0.038203	0.0100	-0.040416	-0.039816
8	0.3000	-0.032958	-0.032419	0.1000	-0.034093	-0.033512	0.0100	-0.035784	-0.035021
9	0.3000	-0.029432	-0.028835	0.1000	-0.030329	-0.029866	0.0100	-0.032062	-0.031289
10	0.3000	-0.026648	-0.025973	0.1000	-0.027678	-0.026952	0.0100	-0.029140	-0.028300
15	0.3000	-0.018357	-0.017485	0.1000	-0.019164	-0.018203	0.0100	-0.020683	-0.019304
20	0.3000	-0.014436	-0.013125	0.1000	-0.015118	-0.013817	0.0100	-0.016334	-0.014771
30	0.3000	-0.010735	-0.008949	0.1000	-0.011366	-0.009414	0.0100	-0.012298	-0.010193
40	0.3000	-0.009206	-0.006712	0.1000	-0.009703	-0.007201	0.0100	-0.010610	-0.007875

TABLE 2

Monte Carlo Estimates of  $E(M, N)$ -CUTOFFS,  $N=1000$   
 NUMBER OF REPLICATIONS=10000

M	ALPHA	$E(M, N)$	ALPHA	$E(M, N)$	ALPHA	$E(M, N)$
1	0.4000	-0.276535	0.1000	-0.292290	0.0100	-0.392535
2	0.4000	-0.135315	0.1000	-0.146162	0.0100	-0.157613
3	0.4000	-0.090789	0.1000	-0.099620	0.0100	-0.109258
4	0.4000	-0.069154	0.1000	-0.076970	0.0100	-0.085471
5	0.4000	-0.056645	0.1000	-0.063591	0.0100	-0.071194
6	0.4000	-0.048560	0.1000	-0.054878	0.0100	-0.061655
7	0.4000	-0.042938	0.1000	-0.048749	0.0100	-0.053375
8	0.4000	-0.038806	0.1000	-0.044359	0.0100	-0.050749
9	0.4000	-0.035909	0.1000	-0.041058	0.0100	-0.047173
10	0.4000	-0.033625	0.1000	-0.038516	0.0100	-0.044410
15	0.4000	-0.028055	0.1000	-0.032237	0.0100	-0.037207
20	0.4000	-0.026820	0.1000	-0.030619	0.0100	-0.035135
30	0.4000	-0.028659	0.1000	-0.032112	0.0100	-0.036311
40	0.4000	-0.032706	0.1000	-0.035957	0.0100	-0.040148
50	0.4000	-0.037506	0.1000	-0.040721	0.0100	-0.044701
250	0.4000	-0.156811	0.1000	-0.161272	0.0100	-0.165923
1	0.3000	-0.260413	0.0500	-0.298241	0.0050	-0.313690
2	0.3000	-0.138010	0.0500	-0.149914	0.0050	-0.169344
3	0.3000	-0.093015	0.0500	-0.102708	0.0050	-0.111870
4	0.3000	-0.071126	0.0500	-0.079688	0.0050	-0.088101
5	0.3000	-0.058290	0.0500	-0.066112	0.0050	-0.073556
6	0.3000	-0.050087	0.0500	-0.057186	0.0050	-0.063835
7	0.3000	-0.044323	0.0500	-0.051053	0.0050	-0.057171
8	0.3000	-0.040243	0.0500	-0.046514	0.0050	-0.052227
9	0.3000	-0.037199	0.0500	-0.043125	0.0050	-0.048037
10	0.3000	-0.034807	0.0500	-0.040479	0.0050	-0.046037
15	0.3000	-0.029124	0.0500	-0.033953	0.0050	-0.038499
20	0.3000	-0.027781	0.0500	-0.032184	0.0050	-0.036544
30	0.3000	-0.029529	0.0500	-0.033442	0.0050	-0.037570
40	0.3000	-0.033477	0.0500	-0.037209	0.0050	-0.040946
50	0.3000	-0.038332	0.0500	-0.042046	0.0050	-0.045758
250	0.3000	-0.157944	0.0500	-0.162971	0.0050	-0.167206
1	0.2000	-0.285486	0.0250	-0.303572	0.0025	-0.363572
2	0.2000	-0.141417	0.0250	-0.153489	0.0025	-0.153489
3	0.2000	-0.095727	0.0250	-0.105685	0.0025	-0.105685
4	0.2000	-0.073571	0.0250	-0.082191	0.0025	-0.082191
5	0.2000	-0.060507	0.0250	-0.066317	0.0025	-0.066317
6	0.2000	-0.052066	0.0250	-0.059304	0.0025	-0.059304
7	0.2000	-0.046198	0.0250	-0.052992	0.0025	-0.052992
8	0.2000	-0.041991	0.0250	-0.048260	0.0025	-0.048260
9	0.2000	-0.038776	0.0250	-0.044829	0.0025	-0.044829
10	0.2000	-0.036416	0.0250	-0.042167	0.0025	-0.042167
15	0.2000	-0.030429	0.0250	-0.035420	0.0025	-0.035420
20	0.2000	-0.028920	0.0250	-0.033492	0.0025	-0.033492
30	0.2000	-0.030084	0.0250	-0.034678	0.0025	-0.034678
40	0.2000	-0.034461	0.0250	-0.038350	0.0025	-0.038350
50	0.2000	-0.039314	0.0250	-0.043144	0.0025	-0.043144
250	0.2000	-0.159343	0.0250	-0.164369	0.0025	-0.164369

TABLE 3

Monte Carlo Estimates of  $H(M, N)$ -CUTOFFS,  $N=10000$   
 NUMBER OF REPLICATIONS=1000

M	ALPHA	H(M, N)	ALPHA	H(M, N)	ALPHA	H(M, N)
1	0.4000	-0.271780	0.1000	-0.276558	0.0100	-0.282290
2	0.4000	-0.131208	0.1000	-0.134215	0.0100	-0.137528
3	0.4000	-0.006597	0.1000	-0.009042	0.0100	-0.091378
4	0.4000	-0.064700	0.1000	-0.066869	0.0100	-0.069138
5	0.4000	-0.051635	0.1000	-0.053676	0.0100	-0.055964
6	0.4000	-0.043078	0.1000	-0.044932	0.0100	-0.046938
7	0.4000	-0.037023	0.1000	-0.038720	0.0100	-0.040416
8	0.4000	-0.032545	0.1000	-0.034093	0.0100	-0.033704
9	0.4000	-0.029083	0.1000	-0.030529	0.0100	-0.032062
10	0.4000	-0.026313	0.1000	-0.027678	0.0100	-0.029140
15	0.4000	-0.018474	0.1000	-0.019164	0.0100	-0.020603
20	0.4000	-0.014175	0.1000	-0.015118	0.0100	-0.016334
30	0.4000	-0.010538	0.1000	-0.011366	0.0100	-0.012298
40	0.4000	-0.009040	0.1000	-0.009788	0.0100	-0.010610
50	0.4000	-0.008346	0.1000	-0.009004	0.0100	-0.009064
250	0.4000	-0.016309	0.1000	-0.017086	0.0100	-0.017630
500	0.4000	-0.031464	0.1000	-0.032089	0.0100	-0.032759
2500	0.4000	-0.153856	0.1000	-0.155276	0.0100	-0.156482
1	0.3000	-0.272069	0.0500	-0.278373	0.0050	-0.202546
2	0.3000	-0.131953	0.0500	-0.135420	0.0050	-0.137804
3	0.3000	-0.087202	0.0500	-0.089921	0.0050	-0.092385
4	0.3000	-0.065211	0.0500	-0.067693	0.0050	-0.069680
5	0.3000	-0.052163	0.0500	-0.054529	0.0050	-0.056263
6	0.3000	-0.043360	0.0500	-0.045370	0.0050	-0.047343
7	0.3000	-0.037487	0.0500	-0.039466	0.0050	-0.040877
8	0.3000	-0.032958	0.0500	-0.034794	0.0050	-0.036188
9	0.3000	-0.029432	0.0500	-0.031164	0.0050	-0.032433
10	0.3000	-0.026648	0.0500	-0.028201	0.0050	-0.029566
15	0.3000	-0.018357	0.0500	-0.019689	0.0050	-0.020795
30	0.3000	-0.014436	0.0500	-0.015572	0.0050	-0.016493
50	0.3000	-0.010735	0.0500	-0.011720	0.0050	-0.012498
100	0.3000	-0.009206	0.0500	-0.010072	0.0050	-0.010854
250	0.3000	-0.008526	0.0500	-0.009335	0.0050	-0.010183
500	0.3000	-0.016727	0.0500	-0.017324	0.0050	-0.017792
2500	0.3000	-0.031602	0.0500	-0.032259	0.0050	-0.032962
1	0.2000	-0.154178	0.0250	-0.155655	0.0050	-0.156909
2	0.2000	-0.274281	0.0250	-0.280814	0.0050	
3	0.2000	-0.132097	0.0250	-0.136606	0.0050	
4	0.2000	-0.087994	0.0250	-0.090697	0.0050	
5	0.2000	-0.063933	0.0250	-0.068416	0.0050	
6	0.2000	-0.052815	0.0250	-0.055066	0.0050	
7	0.2000	-0.044126	0.0250	-0.046288	0.0050	
8	0.2000	-0.038011	0.0250	-0.039990	0.0050	
9	0.2000	-0.033395	0.0250	-0.033252	0.0050	
10	0.2000	-0.029844	0.0250	-0.031508	0.0050	
15	0.2000	-0.027061	0.0250	-0.028708	0.0050	
20	0.2000	-0.018766	0.0250	-0.020088	0.0050	
30	0.2000	-0.014703	0.0250	-0.015932	0.0050	
40	0.2000	-0.010936	0.0250	-0.012028	0.0050	
50	0.2000	-0.009435	0.0250	-0.010347	0.0050	
250	0.2000	-0.008877	0.0250	-0.009539	0.0050	
500	0.2000	-0.016889	0.0250	-0.017492	0.0050	
2500	0.2000	-0.031008	0.0250	-0.032462	0.0050	
2500	0.2000	-0.154662	0.0250	-0.155928	0.0050	

TABLE 4  
ASYMPTOTIC POWER APPROXIMATION EVALUATION  
(ALTERNATIVE : F, ALPHA = .05)

n	10000 Replicates			10000 Replicates			1000 Replicates			10000 Replicates		
	Asymptotic Power Using Simulated Cutoffs	Simulated Power	Asymptotic Power Using Simulated Cutoffs	Simulated Power	Asymptotic Power Using Simulated Cutoffs	Simulated Power	Asymptotic Power Using Simulated Cutoffs	Simulated Power	Asymptotic Power Using Simulated Cutoffs	Simulated Power	Asymptotic Power Using Simulated Cutoffs	Simulated Power
1	0.009544	0.0496	0.031386	0.0499	0.045248	0.0560						
2	0.001974	0.0500	0.021571	0.0519	0.044494	0.0540						
3	0.0000220	0.0501	0.014028	0.0521	0.040716	0.0590						
4	0.000014	0.0508	0.008446	0.0507	0.032091	0.0620						
5	0.0	0.0498	0.004879	0.0525	0.024144	0.0530						
6	0.0	0.0501	0.002911	0.0523	0.022051	0.0560						
7	0.0	0.0505	0.001660	0.0516	0.019433	0.0610						
8	0.0	0.0497	0.001054	0.0528	0.018450	0.0590						
9	0.0	0.0495	0.000643	0.0528	0.017837	0.0610						
10	0.0	0.0501	0.000494	0.0537	0.021225	0.0680						
15	0.0	0.0497	0.000766	0.0527	0.045716	0.0660						
20	0.0	0.0500	0.003139	0.0518	0.095576	0.0650						
30	0.0	0.0498	0.025842	0.0532	0.211254	0.0710						
40	0.0	0.0499	0.05811	0.0531	0.278412	0.0800						

Table 5

m	UNI	KERAND RN2 RN4	RNCG	RN1 SRAND	RN3
1	6.0000	11.00	16.40	12.20	7.40
2	4.8000	6.40	10.20	10.60	10.60
3	5.6000	6.00	6.60	8.00	11.60
4	14.4000	8.00	7.60	10.40	19.20*